Financial Econometrics

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Session 7: Copula Modeling

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Introduction

- In the past session, you have encountered a vast array of financial models
  - Basic ARIMA models for the mean equation
  - GARCH extensions to deal with heteroscedasticity
  - Multivariate GARCH models that deal with dependence modeling
- Theoretical problem arises when we talk about **dependence**
  - Capturing co-movement between financial asset returns with linear correlation has been the staple approach in modern finance since the birth of Harry Markowitz's portfolio theory
  - But linear correlation is only appropriate when the dependence structure (or joint distribution) follow a normal distribution
- Enter copulas - flexible framework to model general multivariate dependence

References

- The work on copulas are vast, as this isn't just a technique but a field of statistics
  - Go to reference: Joe (2014)
  - Financial applications: Ruppert (2011), Carmona (2014)
  - Actuarial: Charpentier (2014)
- For applications on South Africa is limited, but there are some works for emerging markets

Goal

- To introduce to you an extension in the field of risk management to a multivariate space with multiple assets.
- Grasp basic concepts and generators within the field of copulas
  - Learn to walk, before we can run
  - Revisit your statistics
- Understand the basic theory behind copulas, followed by a detailed practical showing how it can be fitted and used in R.

Fields where copulas are applied

- Quantitative finance
  - Stress-tests and robustness checks
  - "Downside/crisis/panic regimes" where extreme downside events are important
  - Pool of asset evaluation
  - Latest development: Vine Copulas
  - Hot research page here
- Civil engineering
- Warranty data analysis
- Turbulent combustion
- Medicine

Introduction to copulas

- Copula stems from the latin verb copulare; bond or tie.
  - Regulatory institutions are under pressure to build robust internal models to account for risk exposure
  - Fundamental ideology of these internal models rely on joint dependency among whole basket of mixed instruments
  - This issue can be addresed through the copula instrument
  - It functions as a linking mechanism between uniform marginals such as the cdf
- Copula theory was first developed by Sklar in 1959 Nelsen (2007).

Introduction to copulas (Sklar)

- Sklar's theorem forms the basis for copula models as:
  - It does not require identical marginal distributions and allows for n-dimensional expansion
- Let X be a random variable with marginal cumulative distribution function:
  - $F_X(x) = \mathcal{P}(X \leq x)$
  - Probability that random variable X takes on a value less or equal to point of evaluation
  - $F_X(x) \sim U(0, 1)$

Introduction to copulas (Sklar cont.)

- If we now denote the inverse CDF (Quantile funtion) as F_x^{-1}
  - $U \sim U(0,1)$ then $F_x^{-1}(U) \sim F(X)$
- This allows a simple way for us to simulate observations from the F_X provided the inverse cdf, F_X^{-1} is easy to calculate
- Think, median is $F_X^{-1}(0.5)$

Lets have a look visually

Definitions and basic properties

- Define the uniform distribution on an interval $(0, 1)$ by $U(0, 1)$, i.e the probability of a random variable U satisfying $P(U \leq u) = u$ for $u \in (0, 1)$
  - This is the quantile function ($Q = F^{-1}$) Probability transformation implies that if X has a distribution function F, then $F(X) \sim U(0, 1)$ iff F is continous

Definitions and basic properties (cont)

Definition (Copula): A d-dimensional copula is the distrubutiom function $\mathcal{C}$ of a random vector U whose components U_k are uniformly distributed

$$C(u_1, \dots, u_d) = P(U_1 \leq u_1, \dots, U_d \leq u_d), (u_1, \dots, u_d) \in (0,1)^d \qquad (1)$$

Thus Sklar's theorem states:

$$\begin{aligned} C(F_1(x_1), \dots, F_d(x_d)) &= P(U_1 \leq F_1(x_1), \dots, U_d \leq F_d(x_d)) \\ &= P(F_1^{-1}(U_1) \leq x_1, \dots, F_d^{-1}(U_d) \leq x_d) \\ &= F(x_1, \dots, x_d) \qquad (2) \end{aligned}$$

Kendall's Tau

- Let (X_1, Y_1) and (X_2, Y_2) be i.i.d random vectors, each with joint distribution function H
- Tau is then defined as the probability of concordance minus the probability of discordance

$$\tau = \tau_{X,Y} = P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2) - (Y_1 - Y_2) < 0) \tag{3}$$

- Tau is the difference between the probability that the observed data are in the same order versus the probability that the observed data are no in the same order

Meeting the generator

The Gaussian copula C_p^{Ga} is a distribution over the unit cube $[0,1]^d$. This represents a linear correlation matrix ρ. C_p^{Ga} is defined as the distribution function of a random vector $(\psi(X_1), \ldots, \psi(X_d))$, where ψ is the univariate standard normal distribution function where $X \sim, N_d(0, \rho)$

$$C_p^{Ga} = \Phi_R \left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\right) \quad (4)$$

where Φ^{-1} is the inverse cumulative distribution function of a standard normal and Φ_R is the joint cumulative distribution

Student T Copula

The student's t-copula $C^t_{\upsilon,\rho}$ of d-dimension can be characterised by parameter $\upsilon \geq 0$ degrees of freedom and linear correlation matrix ρ. The random vector X has a $t^d(0, \rho\upsilon)$ distribution with univariate function

$$C^t_{\upsilon,\rho} = \mathcal{P}(t_\upsilon(X_1) \leq (u_1), \ldots, t_\upsilon(X_d) \leq u_d) \tag{5}$$

$$= t^d_{\upsilon,\rho}(t_\upsilon^{-1}(u_1), \ldots, t_\upsilon^{-1}(u_d)) \tag{6}$$

Archimedean Copulas

- Most common Archimedean copulas admit an explicit formula (Guassian dont)
  - Archimedean copulas are popular because they allow modeling dependence high dimensions
  - Does this with only one only one parameter, governing the strength of dependence.

Illustration

copula fam.	κ_L or λ_L	κ_U or λ_U	refl. sym.	neg. dep.
	1-parameter			
Gaussian	$\kappa_L = 2/(1+\rho)$	$\kappa_U = 2/(1+\rho)$	Y	Y
Plackett	$\kappa_L = 2$	$\kappa_U = 2$	Y	Y
Frank	$\kappa_L = 2$	$\kappa_U = 2$	Y	Y
reflected MTCJ	$\kappa_L = 2$	$\lambda_U = 2^{-1/\delta}$	N	Y*
Joe/B5	$\kappa_L = 2$	$\lambda_U = 2 - 2^{1/\delta}$	N	N
Gumbel (EV)	$\kappa_L = 2^{1/\delta}$	$\lambda_U = 2 - 2^{1/\delta}$	N	N
Galambos (EV)	$\kappa_L = 2 - 2^{-1/\delta}$	$\lambda_U = 2^{-1/\delta}$	N	N
Hüsler-Reiss (EV)	$\kappa_L = 2\Phi(\delta^{-1})$	$\lambda_U = 2 - 2\Phi(\delta^{-1})$	N	N
integ.pos.stableLT	$\kappa_L = 2^{1/\delta}$	$\kappa_U = 2 \wedge (1+\delta^{-1})$	N	Y
inv.gammaLT	$\kappa_L = \sqrt{2}$	$\kappa_U = 1 \vee (\delta^{-1} \wedge 2)$	N	N

Figure: Some families

See Joe (2014)

Vine-Copulas

- A vine is a graphical tool for labeling constraints in high-dimensional probability distributions
- Regular Vines from part of what is know as pair copula construction
- Trees are constructed between copulas based on what is know as maximum spanning degree (Or concordance measure)

Under suitable differentiability conditions, any multivariate density $F_{1 \dots n}$ on n variables may be represented in closed form as a product of univariate densities and (conditional) copula densities on any R-vine V

Vine copulas

The R-vine copula density is uniquely identified according to Theorom 4.2 of kurowicka2006:

$$c(F_1(x_1), \cdots, F_d(x_d)) = \prod_{i=1}^{d-1} \prod_{e \in E_i} c_{j(e),k(e)|D(e)} \left(F(x_{j(e)}|x_{D(e)})\right) \qquad (7)$$

- Introduction to VineCopula
- Website for the research here

References

Carmona, René. 2014. *Statistical Analysis of Financial Data in R*. Vol. 2. Springer.

Charpentier, Arthur. 2014. *Computational Actuarial Science with R*. CRC Press.

Joe, Harry. 2014. *Dependence Modeling with Copulas*. CRC Press.

Nelsen, Roger B. 2007. *An Introduction to Copulas*. Springer Science & Business Media.

Ruppert, David. 2011. *Statistics and Data Analysis for Financial Engineering*. Springer.