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Financial Econometrics

Topic 6:
Multivariate Volatility Models

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DEPARTMENT OF ECONOMICS

Plan…

- Today we extend our analysis into the multivariate plane.
- What this implies is simply that we extend the analysis that we have built on now into incorporating more than one variable into the volatility equation.

Readings

- Tsay (2012,2014) and Ruppert (2011) provide good overviews of MV-GARCH models.
- **<u>Good methodological summaries:</u>**
- For good summaries of the multivariate volatility models:

  Silvenoinen and Terasvirta (2008). *Multivariate GARCH models*

  Bauwens, et al (2006). *MULTIVARIATE GARCH MODELS: A SURVEY*
- Xekalaki chapter 11 also elaborates on the techniques discussed in the class and incorporating it into OX.
- Enders, chapter 3 touches on some of the basics of Multivariate (MV)-Garch techniques .

What is a multivariate extension in the GARCH framework?

- Up to now, we have focussed on the volatility of returns of single series (univariate), which is of great practical importance.
- Today we begin to look at its extension into the multivariate plain – in order to study co-movements between series.
  - Especially considering that the contemporaneous shocks to financial variables in the same market are typically highly correlated.
  - As such – not controlling for the co-variance between the series studied leads to omitted variable bias
- Another great benefit to studying MV-GARCH models is that it allows us to estimate the volatility shock transmissions between series – which might give us a good indication of how shock-driven volatility persistence spills over to other series – very useful in studies determining the extent of shock contagion, e.g. (see e.g. Tse and Tsui, 2002).

What is a multivariate extension in the GARCH framework?

- The first attempts at extending the volatility modelling process into the multivariate framework was intuitively very simple – it basically involved a natural combination of all the univariate systems for each series, as follows:

- Suppose $y_t = \begin{pmatrix} Y_1 \\ \vdots \\ Y_N \end{pmatrix} = (nx1)$ vector of series (let's suppose asset returns), so that $E(y_t) = \mu_t = conditional\ mean\ vector.$
- Then let $\varepsilon_t = y_t - \mu_t = (\mathrm{nx1})$ vector of stochastic residual series, so that the **conditional variance / co-variance** matrix ($\boldsymbol{H_t}$) would then be:

$$H_t = \begin{pmatrix} h_{11,t} & \cdots & h_{1n,t} \\ \vdots & \ddots & \vdots \\ h_{n1,t} & \cdots & h_{nn,t} \end{pmatrix} \qquad \text{and} \qquad \varepsilon_t \sim N(0, H_t)$$

Before we set out… consider:

- We also assume that the residual series can be written as:

$$\varepsilon_t = \sqrt{H_t}.\eta_t, \qquad \text{with} \quad \eta_t \sim N(0, I)$$

- The literature on MV-GARCH is then about fitting the most appropriate variance-covariance matrix $\boldsymbol{H_t}$.
- <u>Before we look at how this is done, first consider that:</u>
  - The **<u>number of parameters</u>** (which can grow large rapidly as n increases!) – trade-off between parsimony and ability of the model to take into account the dynamics in the covariance structure.
  - Also, the matrix needs to be **<u>positive definiteness is required</u>**– should this be implied by the modelling structure of H_t?

Positive Semi-Definiteness of H_t

- With H_t being the variance-covariance matrix, it must hold that H_t is positive definite – else the following could happen:

$$H_t = \begin{pmatrix} 1 & 0.05 & 0.90 \\ 0.05 & 1 & 0.95 \\ 0.90 & 0.95 & 1 \end{pmatrix}$$

Check for yourself that the determinant is negative.

Thus the covariance matrix is not positive definite.

But check for yourself – the above matrix H_t makes no sense if off-diagonals are interpreted as correlations!!)

How we model H_t

- In this course we will be studying two broad ways of fitting H_t :

1. By modelling H_t directly – i.e. direct generalizations of the univariate approach – using the VECH and the BEKK models and its variants.
2. Non-Linear combinations of univariate G+ARCH models – by modelling the conditional variances and conditional correlations separately – i.s.o. modelling the Conditional Covariance matrix directly (this is a more parsimonious method) – we use the Conditional Correlation approaches, and its variants (CCC, DCC, ADCC, etc.)

Other methods (not discussed for brevity) include: Factor Modelling, where we assume ε_t is generated by unobserved heteroskedastic factors; and semi- and non-parametric techniques: which include the Stochastic Volatility (SV)-approach and realized volatility (RV)

VECH model

- The first multivariate extension of the GARCH model was the VECH model of Bollerslev, Engle and Wooldridge (1988), which was basically the natural generalization of the univariate processes.
- In the two-variable case, it basically implies the following:

$$\varepsilon_{1,t} = \sqrt{h_{11t}}.\eta_{1,t} \quad ; \varepsilon_{2,t} = \sqrt{h_{22t}}.\eta_{2,t}$$

h_{11}: variance process of y_1

$$\boldsymbol{h_{11,t}} = c_{10} + \left(\alpha_{11}\varepsilon_{1,t-1}^2 + \alpha_{12}\varepsilon_{1,t-1}\varepsilon_{2,t-1} + \alpha_{13}\varepsilon_{2,t-1}^2\right) + (\beta_{11}h_{11,t-1} + \beta_{12}h_{12,t-1} + \beta_{13}h_{22,t-1})$$

h_{12}: co-variance of y_1 & y_2

$$\boldsymbol{h_{12,t}} = c_{20} + \left(\alpha_{21}\varepsilon_{1,t-1}^2 + \alpha_{22}\varepsilon_{1,t-1}\varepsilon_{2,t-1} + \alpha_{23}\varepsilon_{2,t-1}^2\right) + (\beta_{21}h_{11,t-1} + \beta_{22}h_{12,t-1} + \beta_{23}h_{22,t-1})$$

h_{22}: variance process of y_2

$$\boldsymbol{h_{22,t}} = c_{30} + \left(\alpha_{31}\varepsilon_{1,t-1}^2 + \alpha_{32}\varepsilon_{1,t-1}\varepsilon_{2,t-1} + \alpha_{33}\varepsilon_{2,t-1}^2\right) + (\beta_{31}h_{11,t-1} + \beta_{32}h_{12,t-1} + \beta_{33}h_{22,t-1})$$

VECH model

- In the direct generalization on the previous slide, we basically have the conditional variance process of the two series conforming to the univariate GARCH(1,1) form:

$$h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta h_{t-1}$$

- Now with:

$\varepsilon_{1,t}$ depending on:	$h_{11,t}$ depending on:
Its own lagged squared errors: $(\varepsilon_{1,t-1}^2)$	the past of the variance of y_1 $(h_{11,t-1})$
$y_2's$ lagged squared errors: $(\varepsilon_{2,t-1}^2)$	the past of the variance of y_2 $(h_{22,t-1})$
The product of the lagged squared errors of both y_1 & y_2: $(\varepsilon_{1,t-1}\varepsilon_{2,t-1})$	The conditional covariance between the two series $(h_{12,t-1})$

Notation Note:
$h_{1,t}^2 = h_{11,t}$
And
$cov(y_1; y_2) = h_{12,t}$

VECH model then in standard notation:

- $Vech(H_t) = C + \sum A_j\, vech(\varepsilon_{t-1}\varepsilon'_{t-1}) + \sum B_j\, vech(H_{t-1})$
- With $VECH$ → **stack-operator** that stacks the columns of the lower triangular part of the square matrix; C, A, B → parameter matrices.
- I.e. for the three variable case, this implies:

$$\begin{bmatrix} H_{11,t} \\ H_{12,t} \\ H_{22,t} \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} \varepsilon^2_{1,t-1} \\ \varepsilon_{1,t-1}\varepsilon_{2,t-1} \\ \varepsilon^2_{2,t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} h_{11,t-1} \\ h_{12,t-1} \\ h_{22,t-1} \end{bmatrix}$$

This technique has large demands on parameters → e.g. for three variables it is: 21, and for five variables it is already 465 parameters!!

Diagonal VECH

- To counter the large amounts of parameters – we can use the **diagonalised** version of VECH (D-VECH) – with the obvious downside being the it does not accurately account for the dynamics of the conditional correlations.
- Thus the three variable DVECH(1,1) model is:

$$\begin{bmatrix} H_{11,t} \\ H_{12,t} \\ H_{22,t} \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t-1}^2 \\ \varepsilon_{1,t-1}\varepsilon_{2,t-1} \\ \varepsilon_{2,t-1}^2 \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix} \begin{bmatrix} h_{11,t-1} \\ h_{12,t-1} \\ h_{22,t-1} \end{bmatrix}$$

- Which yields conditional variances for each series that is the same as the univariate GARCH process ($h_{11,t} = \omega_1 + \alpha_{11}\varepsilon_{1,t-1}^2 + b_{11}h_{11,t-1}$), while the conditional covariance is also simply ($h_{12,t} = \omega_2 + \alpha_{22}\varepsilon_{1,t-1}\varepsilon_{1,t-1}{}' + b_{22}h_{12,t-1}$)
- Note that this model, although less parameter hungry – has no interaction between the different conditional variances and covariances and is often considered too restrictive in this regard.

Riskmetrics

- The Riskmetrics model (1996) developed by JP Morgan, uses an exponentially weighted moving average model (EWMA) to forecast variances and covariances, which in the two-variable case looks like:

$$h_{ij,t} = (1-\lambda)\varepsilon_{i,t-1}\varepsilon_{j,t-1} + \lambda h_{ij,t-1}$$

Which is a **Scalar VECH** model (where the parameter matrices A&B are now scalars $(1-\lambda)$ & λ.

- The decay factor λ is suggested to be 0.94 for daily data and 0.97 for monthly.
- This makes the model easy to work with (and hence preferred by practitioners) – implying no need for parameter estimation…
- But not so easy to motivate empirically when applied to different data types.

BEKK-GARCH

- The Baba, Engle, Kraft & Kroner (BEKK) MV-GARCH extension to the VECH model essentially imposes the positive definiteness condition on the $H_t - matrix$ as follows (for the bivariate case):

$$H_t = C'C + A\varepsilon_{t-1}\varepsilon'_{t-1}A' + B'H_{t-1}B$$

Which implies that the parameter matrices are effectively squared, ensuring that the Covariance matrix H_t is positive definite. In the 3 variable case:

$$C = lower\ triangular; \quad A\&B = 3\text{x}3\ matrix$$

- This approach, although ensuring positive definiteness, can again be burdensome in terms of the amount of parameters needed to be specified.

BEKK Model

- From the BEKK model, the matrix A shows how conditional variances correlate with **shocks** or news **impacts** – measured by the past squared errors (ε_t^2)
- The matrix B shows how past conditional variance affects contemporary conditional variance (i.e. persistence in conditional variance / volatility momentum)
- The bi-variate BEKK model written out is:

$$\varepsilon_{1,t} = \sqrt{h_{11t}}.\eta_{1,t} \quad ;\varepsilon_{2,t} = \sqrt{h_{22t}}.\eta_{2,t}$$

$$\boldsymbol{h_{11,t}} = c_{11}^2 + \left(\alpha_{11}^2\varepsilon_{1,t-1}^2 + 2.\alpha_{11}\alpha_{12}\varepsilon_{1,t-1}\varepsilon_{2,t-1} + \alpha_{21}^2\varepsilon_{2,t-1}^2\right)$$

$$+(\beta_{11}h_{11,t-1} + 2\beta_{11}\beta_{12}h_{12,t-1} + \beta_{21}h_{22,t-1})$$

$$\boldsymbol{h_{12,t}} = c_{12}^2 + \left(\alpha_{11}\alpha_{12}\varepsilon_{1,t-1}^2 + (\alpha_{21}\alpha_{21} + \alpha_{11}\alpha_{22})\varepsilon_{1,t-1}\varepsilon_{2,t-1} + \alpha_{21}\alpha_{22}\varepsilon_{2,t-1}^2\right)$$

$$+\cdots$$

$$\boldsymbol{h_{22,t}} = \cdots$$

- (proofs of efficiency are in Engle and Kroner (1995))

BEKK Model

- Some of the most useful restrictions on the BEKK model include: the Diagonal BEKK model and the Scalar BEKK model.
- Diagonal-BEKK model: under the full-BEKK model on the previous slide, each parameter matrix A and B require n^2 parameters to be estimated.
- The Diagonal BEKK model restricts these matrices to be diagonal (with it being a more parsimonious model than the DVECH).

- Scalar BEKK model further reduces the parameter burden by assuming the same values hold for each element in A & B, thus $A = aI$ (with $a = scalar$)

Use of these multivariate models

- The main use of these models is in studying volatility spill-over effects. See Hassan & Malik's paper who use a tri-variate full BEKK model to study volatility spill-overs between US sector indices (note the expansion of the tri-variate BEKK model on page 474).
- They use the tri-variate BEKK model to test whether there are any significant volatility spill-over effects from **news** (shock effect) emanating from other sectors, by viewing the parameters for $(\varepsilon_{1,t}\varepsilon_{2,t})$& $(\varepsilon_{1,t}\varepsilon_{3,t})$, e.g., in the $h_{11,t}$ equation (which would identify volatility spill-overs from news events from sector $2 \rightarrow 1$ and sector $3 \rightarrow 1$, respectively)
- And also from **direct volatility spill-overs** by viewing parameters for $h_{12,t}$ & $h_{13,t}$ (identifying volatility spill-overs from other sectors' volatility, from sector $2 \rightarrow 1$ and sector $3 \rightarrow 1$, respectively)

BEKK(1,1)-GARCH-in-Mean model?

- Of course we can add a GARCH-in-mean extension to the MV-GARCH process being studied, to account for the risk premium in terms of the mean equation too.
- The individual univariate GARCH specifications we did earlier can also be used in specifying the univariate processes: $h_{11,t}$, $h_{22,t}$.

Some notable studies…

- Horvath study the spill-over effects of several countries' aggregate share index (including SA)
- {Another good example is Christopher, et al: *Do sovereign credit ratings influence regional stock and bond market interdependencies in emerging countries?*}
    - They use a bi-variate BEKK model to study spill-overs
    - The authors also fit the dynamic conditional correlations between each share market pair, defined as:

$$p_{12,t} = \frac{h_{12}}{\sqrt{h_{11,t}h_{22,t}}}$$

This allows us to get an estimate of the time-varying conditional correlation…

- Hassan and Malik use a Tri-variate GARCH model to study spill-over effects between the six largest US sectors. This paper discusses the methodology very clearly.

  Due to the high amount of parameters and difficulty in estimation of higher order GARCH models, authors typically only do **tri-variate** BEKK GARCH models.

- Beirne, *et al* use a tri-variate BEKK-GARCH-(1,1)-in-mean model to study the spill-over effects of 41 EMEs in Asia, EU, South America and the Middle East.
  - Their use of a **full-trivariate BEKK** model requires a large amount of parameters to be estimated!

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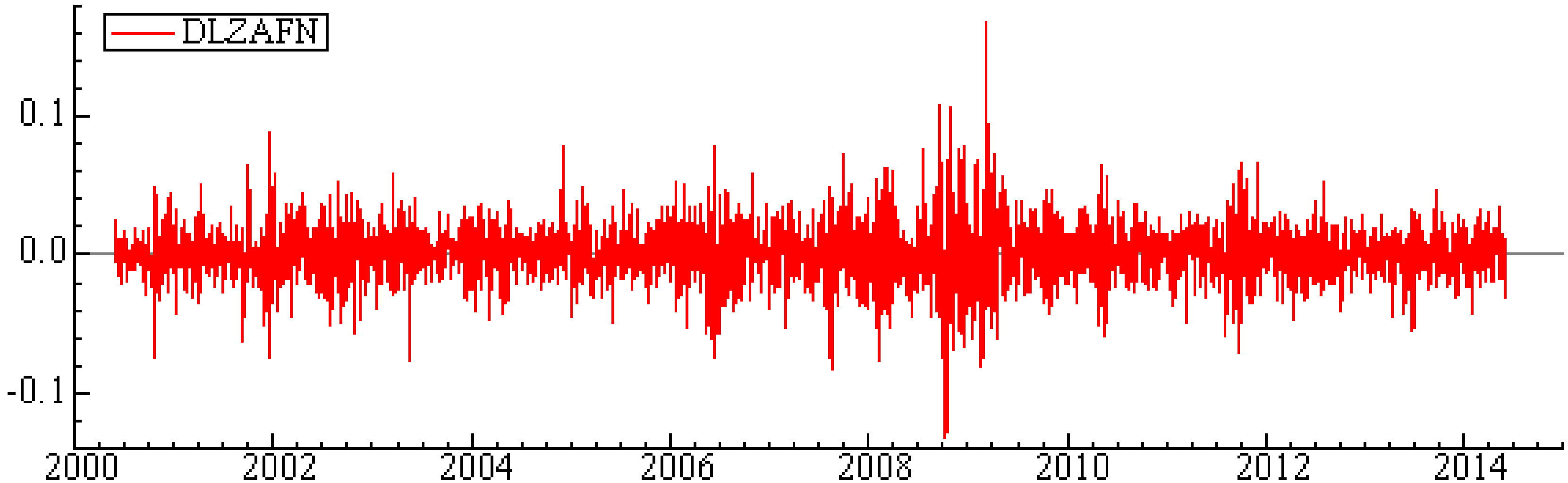
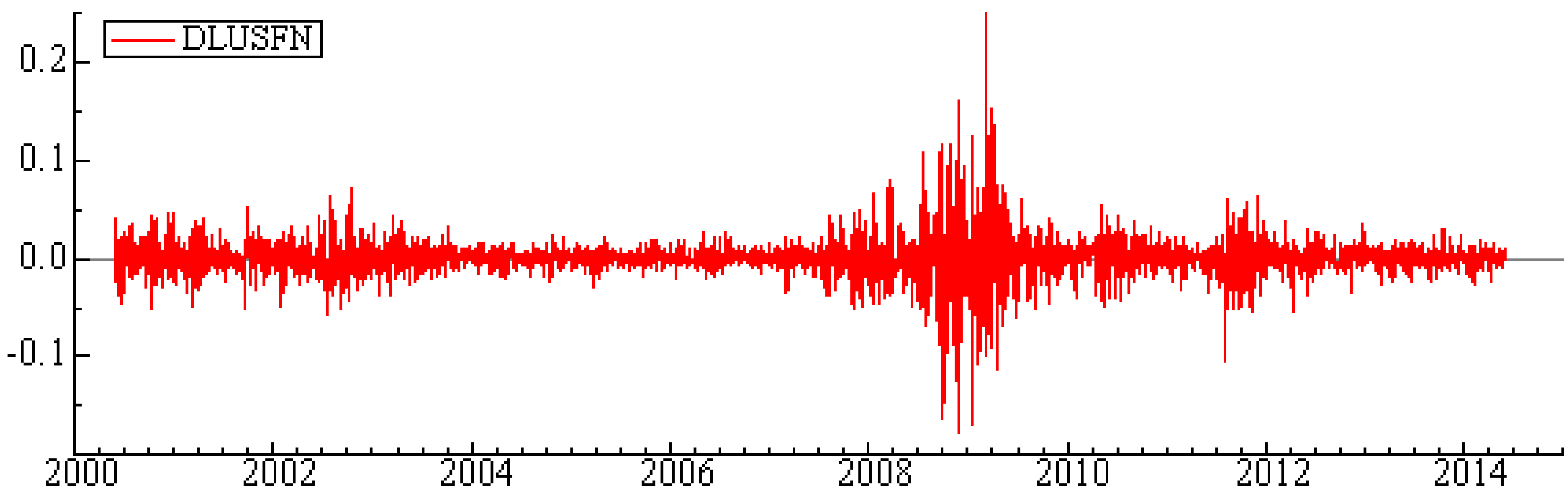
Dynamic Conditional Correlation Models

Department of Economics

DEPARTMENT OF ECONOMICS

Motivation: Daily Returns to ZA and US Financial sectors – display periods of high and low correlations – **implication on diversification?**





Why Dynamic Conditional Correlations?

- Note from the calculated conditional correlations using the BEKK models – the variances and covariances are calculated separately, with the correlations inferred at the end.
- Engle, et al (2002) propose modelling the correlations as a dynamic, time-varying process **directly.**
- Doing so requires estimation of the conditional correlation matrix directly… and modelling univariate volatility estimates separately (2-step approach).
- This leads to gains in parsimony and a more direct approach to fitting dynamic correlations (as noted, BEKK and VECH suffer from parameter dimensionality, and becomes infeasible when $k > 3$).
- It also allows flexibility i.t.o. specifying the univariate models.

CCC model

- Bollerslev (1990) was the first to propose the class of models that does what we set out to do in the previous few slides.
- This was the CCC (constant conditional correlation) model – where the conditional correlations are kept constant over time.
- The main benefit of this approach is that it conserves parameter usage greatly (parsimonious benefits) and also simplifies the estimation procedure.
- The model looks as follows:

CCC model

$$Dlog\left(Y_{i,t}\right) = r_{it}$$

$$r_{it} = \mu_{it} + \varepsilon_{it}$$

- $\mu_{it} \rightarrow conditional\ mean\ eq$;
- $\varepsilon_{it} \rightarrow condtionally\ heteroskedastic\ error\ series$

$$\varepsilon_{it} = \sqrt{H_{it}}.\eta_i \ \ ,with\ \varepsilon_{it} \sim N(0, H_t)\ \&\ \eta_i \sim N(0, I)$$

This is the same as last week.

The difference now comes in with how Bollerslev defined the $H_t - matrix\ldots$

CCC model

- Under the CCC-model, the H_t matrix is defined as:

$$H_t = D_t R D_t$$

With:

- $\boldsymbol{D_t} = diag(\sqrt{h_{11t}}, \dots, \sqrt{h_{NNt}})$
- $\boldsymbol{h_{11t}} \rightarrow$ functional form (GARCH, EGARCH, TARCH, …) of the univariate model describing the conditional variance process of Y_1.
- $\boldsymbol{R_{ij}} \rightarrow$ is a positive definite symmetric matrix (with ones on the diagonal) that describes the conditional correlations between the series Y_i & Y_j by the off-diagonal elements ρ_{ij} $(i \neq j)$.

CCC model

- This can be written in matrix form as:
- $H_t = D_t R D_t =$

$$\begin{bmatrix} \sqrt{h_{11}} & 0 & 0 \\ 0 & \sqrt{h_{22}} & 0 \\ 0 & 0 & \sqrt{h_{33}} \end{bmatrix} \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix} \begin{bmatrix} \sqrt{h_{11}} & 0 & 0 \\ 0 & \sqrt{h_{22}} & 0 \\ 0 & 0 & \sqrt{h_{33}} \end{bmatrix}$$

- Or for the bi-variate case written out as:

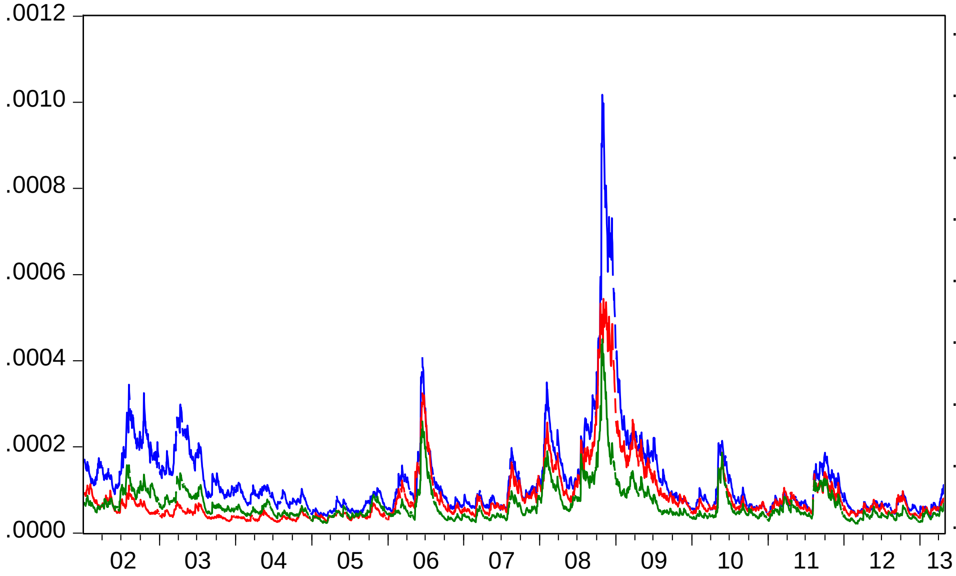
$$h_{11t} = c_1 + \alpha_1 \varepsilon_{11,t-1} + \beta_1 h_{11,t-1}$$

$$h_{12t} = \rho_{12} \sqrt{h_{11,t}} \cdot \sqrt{h_{22,t}}$$

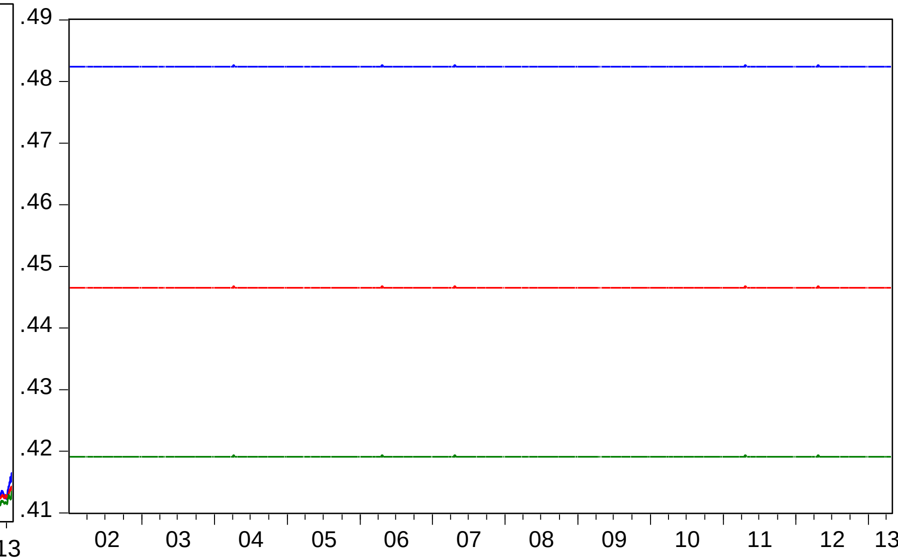
$$h_{22t} = c_2 + \alpha_1 \varepsilon_{22,t-1} + \beta_1 h_{22,t-1}$$

CCC Output

- Note that from the output of the tri-variate CCC– we have time-varying co-variance and variance series, but we assume **constant conditional correlations**…



Conditional covariances



Conditional correlations (ρ_{ij})

Intuitively: what is going on?

- Note that what the CCC model effectively does is it assumes that the conditional **variances** of each series follow univariate GARCH models, while the conditional **covariances** are calculated by assuming the conditional **correlations** are kept constant.
- Thus it separates the calculation of D_t (variance) and H_t (variance-covariance), by assuming constancy of R.
  - This assumption of Conditional Correlation implies far less parameters are used in the process – but with the loss of the time-varying $\boldsymbol{\rho_{12}}$ series we were able to calculate for the BEKK and VECH models.
- **THUS: The dynamics of the covariance is only determined by the dynamics of the two conditional variances…**

DCC – relaxing the constancy of the conditional correlations

- Despite the simplification benefits – this approach is not as intuitively appealing, as <u>assuming constant correlations over time is a strong assumption.</u>
- In response, Engle (2002) showed that the CCC model can be generalized so as to <u>**allow correlations between series to vary over time**</u>, while keeping the benefit of the CCC's simplified calculation of univariate variances.
- <u>In order to do so, we employ a two-step approach by:</u>

1. Obtaining GARCH estimates of the univariate volatility models for each series (using whichever form – GARCH, EGARCH, …).
2. Using the **standardized residuals** (η_t), extracted in step one, to estimate time-varying (dynamic) conditional correlations using a log-likelihood approach

DCC Model

- In step 1 we fit the univariate GARCH process for each series – to obtain the conditional variances used to standardize the residuals as follows:

$$\eta_{i,t} = \varepsilon_{i,t} / \sqrt{h_{ii,t}}$$

- Then we use these standardized residuals in the second step to make time-varying conditional correlations, done as follows:
- First assume as before:

$$H_t = D_t R_t D_t$$

With $R_t =$ time varying conditional correlations now.

Calculating these dynamic correlations then requires us to assume the following GARCH form to be fitted on the <u>unconditional variance matrix</u>:

DCC Model

- In step 1 we fit the univariate GARCH process for each series – to obtain the conditional variances used to standardize the residuals as follows:

Returns series' residuals ($\varepsilon_t = Y_t - \mu_t$)

Standardized (WN) Vol equation resids

$$\eta_{i,t} = \varepsilon_{i,t} / \sqrt{h_{ii,t}}$$

Series i's univariate GARCH model, e.g. GARCH(1,1)

- Then we use these standardized residuals in the second step to make time-varying conditional correlations, done as follows:
- First assume as before:

$$H_t = D_t R_t D_t$$

With $R_t =$ time varying conditional correlations now.

Calculating these dynamic correlations then requires us to assume the following GARCH form to be fitted on the <u>unconditional variance matrix</u>:

DCC Model

Shock: unconditional - error   Conditional Variance Process

$$Q_{ij,t} = \overline{Q_{ij}} + \alpha(\eta_{i,t-1}\eta'_{j,t-1} - \overline{Q_{ij}}) + \beta(\mathrm{Q}_{ijt-1} - \overline{Q_{ij}})$$

So that:

$$Q_{ij,t} = (1-\alpha-\beta).\overline{Q_{ij}} + \alpha(\eta_{i,t-1}\eta'_{j,t-1}) + \beta(Q_{ijt-1})$$

<u>Note this has a **GARCH-form**, with :</u>

$$\eta_{i,t-1} = \varepsilon_{i,t-1}\,\sigma_{i,t-1}^{-1} \rightarrow \text{Standardised residuals from step 1}$$

Q_{ij} → **conditional variance** & $\overline{Q_{ij}}$ → the **unconditional covariance** of the standardized residuals estimated in step 1.

As before we have restrictions $\alpha + \beta < 1$ and each > 0.*

* (for a more detailed discussion of the math, see Engle (2002). Dynamic conditional correlation – a simple class of MV-GARCH).

DCC Model

- Because we estimated the unconditional variance and covariance processes in step 1 – all we now need to do is estimate the parameters α & β using MLE.

- Then we can derive the dynamic (time-varying) conditional correlation matrix, R_t, as follows:

$$R_t = {Q^*_{ii,t}}^{-1} . Q_{ij,t} . {Q^*_{jj,t}}^{-1}$$

With $Q^*_{ij,t} \rightarrow$ the diagonal matrix with entries along the diagonal equal to the squared diagonal of $Q_{ij,t}$, or $Q^*_{ij,t} = diag(Q_{ij,t})$

(resembling, if you like, the D_t matrix with $h_{ii,t}$ as diagonals, only now it is $q_{ii,t}$)

DCC Model

- Thus from the previous slide, what we are doing is fitting the R_t matrix having the following elements:

$$\rho_{ij,t} = \frac{q_{ij,t}}{\sqrt{q_{ii,t}.q_{jj,t}}}$$

$$= \frac{(1-\alpha-\beta\ )\bar{q} + \alpha.\eta_{i,t-1}\eta'_{j,t-1} + \beta.q_{ij,t-1}}{\sqrt{((1-\alpha-\beta)\bar{q_i} + \alpha.\eta^2_{i,t-1} + \beta.q_{ii,t-1}))((1-\alpha_1-\beta)\bar{q_j} + \alpha.\eta^2_{j,t-1} + \beta.q_{jj,t-1}))}}$$

Note that R_t has diagonals$= 1$ and off-diagonals $=$ conditional correlations

<u>This is done using Log-likelihood maximization of the function</u>.

This process (as noted in Engle (2002)) is consistent in its two step approach, and greatly reduces the amount of parameters as $N \to \infty$

DCC Model

- As suggested in Bauwens, et al (2006: 90) a drawback of the DCC approach is that it assumes the dynamic correlation process to behave the same over time in reaction to past shocks.
- For the equation of $Q_{ij,t}$ - note that the structure is constant over time.
- This might not be the case (i.e. α & β might well change over time) and can be considered a serious drawback to the approach.

- Another drawback is that the equation for $Q_{ij,t}$ does not differentiate between positive and negative shocks. I.e. it has no built-in leverage function. This is addressed in the ADCC model

Flexibility…

- As the DCC estimation procedure uses an efficient 2-step procedure, it is insensitive to the specification of the univariate approach used.
- The only input into step 2 from step 1 (other than the sample / unconditional covariance estimates, $\bar{Q}$) is the standardized residuals.
  - Thus we can choose any univariate GARCH specification, and control for spill-overs as well – as long as we have cleaned standardized residuals as input (η_i)

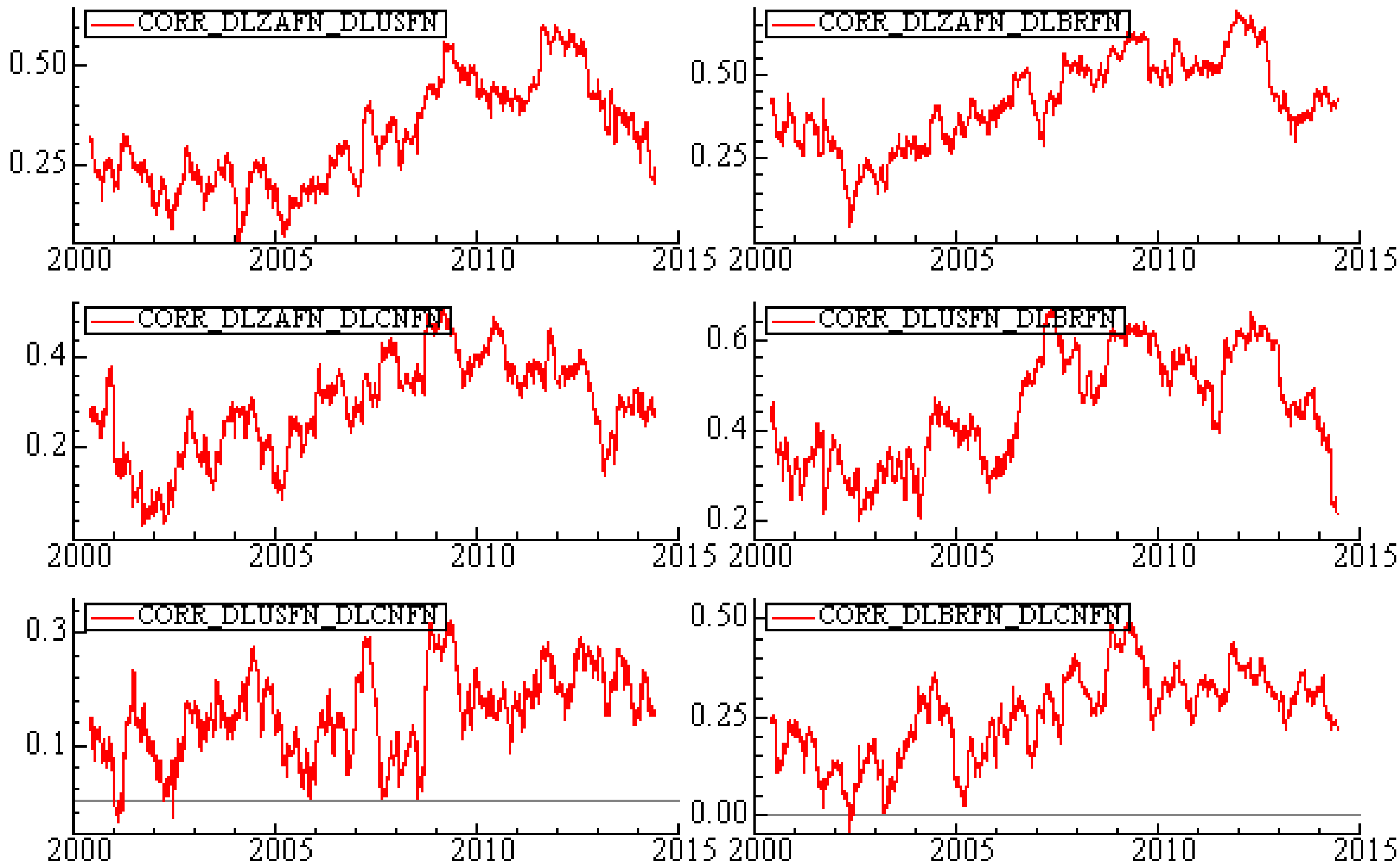
DCC model in R

- # First specify the univariate normal GARCH(1,1) for each series
- > garch11.spec = ugarchspec(mean.model = list(armaOrder = c(0,0)),
- + variance.model = list(garchOrder = c(1,1),
- + model = "sGARCH"),
- + distribution.model = "norm")

- # Then do the dcc specification - GARCH(1,1) for conditional correlations
- > dcc.garch11.spec = dccspec(uspec = multispec(replicate(2,
- garch11.spec)),
- + dccOrder = c(1,1),
- + distribution = "mvnorm")

- > dcc.fit = dccfit(dcc.garch11.spec, data = MSFT.GSPC.ret)
- > slotNames(dcc.fit)
- [1] "mfit" "model"
- > names(dcc.fit@mfit)
- > names(dcc.fit@mfit)
- > plot(dcc.fit)

Graphical representation of DCCs



ADCC Model

- Cappiello, et al (2006) proposed the introduction of **leverage** effects into the DCC model.
- This was done by extending the $Q-$ equation as follows:

$$Q_{ij,t} = (1-\alpha-\beta).\bar{Q} - g(\overline{W_t}) + \alpha\left(\eta_{i,t-1}\eta'_{j,t-1}\right) + \beta\left(Q_{ij,t-1}\right) + g(\xi_{i,t-1}\xi'_{j,t-1})$$

Where:

$$\xi_{i,t-1} = 1.\eta^2_{i,t} \quad if\ \eta_{i,t} < 0, \qquad zero\ otherwise$$

$W_t = covariance\ of\,(\xi_{i,t-1}\ \xi'_{j,t-1})$ using sample analogue, thus:

$$E\left(\xi_{i,t-1}\ \xi'_{j,t-1}\right) \approx \frac{1}{n}\sum(\xi_{i,t-1}\ \xi'_{j,t-1})$$

ADCC Model

- From the ADCC model, we can then interpret the coefficient g, which is the asymmetric response of the conditional correlation series following **negative** shocks.
- Typically, financial returns display a **significant increase** in their correlations during negative return events – which limits the diversification potential during system wide downward asset price adjustments.

Robust Standard Errors (Sandwich formula)

		Coefficient	Std.Error	t-value	t-prob
	rho_21	0.209297	0.13120	1.595	0.1108
	rho_31	0.336926	0.11220	3.003	0.0027
	rho_41	0.173471	0.13599	1.276	0.2022
	rho_32	0.341678	0.13758	2.484	0.0131
	rho_42	0.011264	0.13987	0.08054	0.9358
	rho_43	0.131234	0.14172	0.9260	0.3545
α	alpha	0.008441	0.0033325	2.533	0.0114
β	beta	0.989308	0.0052995	186.7	0.0000
γ	gamma	0.002828	0.0023063	1.226	0.2202

No. Observations : 3275 No. Parameters : 29

No. Series : 4 Log Likelihood : 34607.316

Elapsed Time : 2.028 seconds (or 0.0338 minutes).

Summary of the MVGARCH models

- The variance-covariance matrix is given as:

$$H_t = D_t R_t D_t$$

- <u>CCC model</u>: $R_t = R$

- <u>DCC model</u>: $H_t = D_t R_t D_t \quad ; \quad D_t = Q_t^* = diag\left(\sqrt{Q_t}\right)$

$$R_t = {Q_{ij,t}^*}^{-1} . Q_{ij,t} . {Q_{ij,t}^*}^{-1} \qquad ;$$

$$Q_{ij,t} = (1 - \alpha - \beta). \overline{Q_{ij}} + \alpha\left(\eta_{i,t-1}\eta'_{j,t-1}\right) + \beta\left(Q_{ijt-1}\right)$$

- <u>ADCC model:</u> Adds $g(\xi_{i,t-1}\xi'_{j,t-1})$ to the Q_{ijt} equation

Orthogonal-GARCH (O-GARCH)

- O-GARCH combines the insights that we learnt from PCA analysis to reduce the dimensionality of a MV-GARCH estimation procedure…
- In particular, it assumes the observed data is generated by an orthogonal transformation of N-univariate GARCH processes.
- The data is then linearly transformed using the orthogonal matrix of eigenvectors of the population of the sample unconditional covariance matrix of the standardized returns series (Note – factor models can also be used as orthogonalizing the system)

Orthogonal-GARCH (O-GARCH)

- The O-GARCH$(1,1,k)$ is then defined as:

$$Y_t = \mu_t + \epsilon_t$$

$$\epsilon_t = V^{1/2}.u_t$$

$$u_t = Z_k f_t$$

With $V_t \rightarrow diag(v_1, \ldots v_N)$, the population variance of ϵ_t

$$Z_k = P_k.L_k^{1/2} = P_k.diag(l_1^{1/2}, \ldots, l_k^{1/2})$$

$l_1 > \cdots > l_k \rightarrow$ the k largest **eigenvalues** of the population correlation dispersion matrix of ϵ_t

$P_k \rightarrow$ the associated **eigenvectors** of the k $-$eigenvalues

Orthogonal-GARCH (O-GARCH)

- Then:

$$E_{t-1}(f_t) = 0 \;\; \& \;\; Var_{t-1} = \sum t = diag(\sigma^2_{f_{1,t}}, \dots, \sigma^2_{f_{k,t}})$$

And

$$\sigma^2_{f_{i,t}} = (1 - \alpha_i - \beta_i) + \alpha_i f^2_{t-1} + \beta_i \sigma^2_{f_{t-1}}, \qquad \forall\, i = 1, \dots k$$

Then:

$$H_t = Var_{t-1}(\epsilon_t) = V^{1/2}\, V_t\, V'^{1/2}$$

With

$$V_t = Var_{t-1}(u_t) = Z_k \sum t\, Z_k'$$

Orthogonal-GARCH (O-GARCH)

- We choose k by applying PCA on the standardized residuals (With V calculated by taking the sample (unconditional) variance estimate) –this we can do by viewing a scree plot, e.g.

- It effectively estimates $k -$ univariate GARCH(1,1) {or any other univariate form} models in order to reduce the dimensionality of the system.

Orthogonal-GARCH (O-GARCH)

- O-GARCH thus involves 3 steps:

1. Calculating the conditional mean equation (may include explanatory factors or be ARFIMA)
2. Compute P_k & L_k by means of PCA
3. Fit k − GARCH-type models on $f_{i,k}$ using QMLE.

GO-GARCH

- Van der Weide propsed a generalized form of the O-GARCH model, that relaxes the orthogonality condition of the previous, by assuming Z is square and invertible, i.s.o. necessarily orthogonal, and is not restricted to be triangular.
- Thus he proposes:

$$Z_k = P_k L_k^{1/2} \, U$$

with $U \rightarrow orthogonal, U = \Pi \, G_{ij}(\delta_{ij}) \, , with \, G_{ij}(\delta_{ij})$ performing rotations in the (i,j) $-$plane ($G \rightarrow$ matrix of sinus and cosinus functions)

The O-GARCH is a special case where $U = I_k$

GO-GARCH-NLS

- See Boswijk and van der Weide (2006) for their extension of the GO-GARCH by using NLS as opposed to QMLE

GOGARCH Presentation

- I uploaded a presentation on work that I did using GOGARCH models and a large panel set of bivariate EM conditional correlations and what drives it.

- The slides contain more of the technicalities of the GOGARCH techniques.

Check the Literature

- <u>Other readers include:</u>
- Antoniou & Pescetto (2007) who combine **DCC** estimates and **BEKK** estimates to identify conditional correlations and conditional volatility spill-overs, respectively, across international stock markets and industrial sectors.
- Johannson uses, instead of a two-step approach, a joint estimation of the DCC. He also fits an ADCC model.
  - Interestingly his mean equation also is a VECM of the series.
  - This paper has some really good definitions and is a good paper to get tips on your methodology.
  - His interpretations of the coefficients are also very clear.

Right… Back to the practical class!