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Financial Econometrics

Topic 1:

Basic Statistical Properties of Financial Returns

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DEPARTMENT OF ECONOMICS

What we will be discussing today

- Today we look at certain **stylized facts** of financial market returns, and how we typically measure it in financial econometrics.
- We will also be looking at some of the common matrix notations and techniques often encountered in the financial econometrics literature.

Reminder of some matrix concepts

- Matrix algebra concepts are important for understanding some of the most fundamental insights in financial econometrics.
- As such, the course will start with a very brief overview of some of the most relevant concepts in matrix notation.

What we know about matrices

- For any matrix: $A = \begin{bmatrix} a & \cdots & f \\ \vdots & \ddots & \vdots \\ e & \cdots & z \end{bmatrix}_{RxC}$
- Square Matrix: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ = (R=C)
- Symmetric Matrix: $A = \begin{bmatrix} a & 2 & 3 \\ 2 & e & 7 \\ 3 & 7 & i \end{bmatrix}$ = (b=d, g=c, h=f)
- Diagonal Matrix: $A = \begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix}$ = (off-diagonals = 0, diagonal $\neq$ 0)
- Identity Matrix: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ => $BA = AB = BI = B$, ∀ matrix B

What we know about matrices

- **Multiplication of a matrix**: $A_{\mathrm{RxC}} B_{C\mathrm{xM}} = \mathrm{C}_{\mathrm{RxM}}$

e.g. → $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} . \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} (1*5)+(2*7) & (1*6)+(2*8) \\ (3*5+4*7) & (3*6+4*8) \end{bmatrix}$

- **Determinant of a matrix:**

- If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow \ \det(\mathrm{A}) = |\mathrm{A}| = 1*4-2*3 = -2.$

Determinants can be found on square matrices. The determinant of A is therefore a number. If $\det(A) \neq 0$, then A has a unique solution, is non-singular and can thus be inverted.

If $Det(A) = 0$, there are either many solutions or none at all.

What we know about matrices

- **Transpose**: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$; $A' = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$
- **Rank** (A) → # **linearly independent rows** in the matrix (also equal to number of linear indep columns…)

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow Rank(A) = 2$ (as row 1 cannot be expressed as a linear multiple of row 2).

$A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \rightarrow Rank(A) = 1$ (as row 1 can be expressed as a linear multiple of row 2 → $R2 = 4 * R1$

The Rank is thus a very useful way of determining whether all the variables (rows) in a model are independent. Remember OLS requires $x_1, \dots x_N$ to be independent if they are exogenous variables…

Thus we can evaluate this by testing if $Rank(X)_{RxR} = R$

What we know about matrices

If $A = \begin{bmatrix} R_{1,1} & \cdots & R_{1,20} \\ \vdots & \ddots & \vdots \\ R_{10,1} & \cdots & z_{10,20} \end{bmatrix}_{10x20}$

- If A is of full Rank, it implies that $Rank(A) = 10$, which implies that all the variables included $(R_1 \rightarrow R_{10})$ can be considered **linearly independent** (if variables → rows).
- If $\boldsymbol{Rank}(A) < 10 \rightarrow$ the opposite holds and A is considered **singular (Singularity of $\boldsymbol{A}$ implies no matrix $\boldsymbol{B}$** exist such that $AB = BA = I$, **i.e. $\boldsymbol{A}$ has no inverse**)

- The **inverse** of a matrix is denoted as: $\boldsymbol{A^{-1}}$
- So that: $A^{-1}A = AA^{-1} = I.$
- A **can only** be inverted if A is **non-singular** (has full rank), and is $\mathbf{square}$ $(R = C)$

What we know about matrices

- If $A = \begin{bmatrix} D_1 & a & b \\ c & D_2 & f \\ d & e & D_3 \end{bmatrix}$, the **Trace** of A is: $Trace(A) = D_1 + D_2 + D_3$

- **Eigenvalues**:

  For any Square matrix A_{RxR} and scalar c with $c_{\mathrm{Rx1}} \neq 0$:

  We can write $Ac = \lambda c$, with λ →set of **scalars**.

  This can of course be rewritten as:

$$Ac = \lambda \mathrm{I}_p c \quad \rightarrow \quad (A - \lambda I_p)c = 0$$

For the system: $(A - \lambda I_p)c$ to have a non-zero solution, it would require $(A - \lambda I_p)$ to be non-singular, because $c \neq 0$.

This can be tested by taking the determinant: $|A - \lambda I_P| = 0$

Here we call λ the characteristic root of the system.

Example of Eigenvalues

- For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \rightarrow \quad |\mathrm{A} - \lambda I_P| = \left| \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$

$$= \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = (1-\lambda) * (4-\lambda) - 6 = \lambda^2 - 5\lambda - 6$$

$$= (\lambda - 2)(\lambda - 3)$$

Thus the solution to the $|\mathrm{A} - \lambda I_P|$ system in this case is:

$$\lambda = 2 \ \& \ \lambda = 3$$

Thus the Eigenvalues of the system is $2 \ \& \ 3$.

Similarly, suppose we have $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$, clearly Row 1 is not linearly independent of Row 2 ($2 * R_1 = R_2$). You can check that the eigenvalues then are: $\lambda = 0$ and $\lambda = 4$. Because one of the eigenvalues is zero, it implies the matrix is not of full rank and thus the rows not all linearly independent (i.e. $B = singular$).

So what makes this special?!

Eigenvalues

- Firstly, if all the **eigenvalues** are **non-zero** (like in the example), it implies that the **matrix** is of **full-rank** and thus **non-singular**.
- Also interesting is that the sum of the eigenvalues is the trace of the matrix, while the product of the eigenvalues is the determinant.
- Eigenvectors can then be thought of as the values of $\boldsymbol{c}$ corresponding to the eigenvalues.
- This implies that for any non-zero vector $\boldsymbol{v}$: $\boldsymbol{Av} = \boldsymbol{\lambda v}$
- Thus v is an eigenvector of A if Av is a scalar multiple of v.
- E.g. $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$, corresponding to the eigenvalue of $\lambda = 3$ since: $A\boldsymbol{x} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \mathbf{3}.x$

What we know about matrices

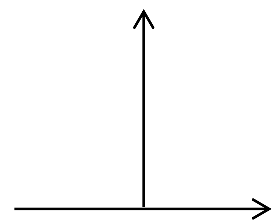
Equivalent statements of matrices can be summarized as:

If A is a nxn Matrix, then:

- $Ax = 0$ can only be solved when $x = 0$
- $Ax = b$ is consistent for every b and has only one unique solution for every nx1 vector b (*so that betas are unique in a regression*)
- $\det(A) \neq 0$
- Column and row vectors of A are all linearly independent
- A has full rank (=n) and is thus non-singular.
- $\lambda = 0$ is not an eigenvalue of A.
- The eigenvectors of A correspond to a linearly independent set of vectors (*thus an implicit form of orthogonalization is achieved!*)

Orthogonality, Orthonormality

- Orthogonality implies in vector form that two vectors are perpendicular:



- From this it is clear that the two lines are **independent** & **uncorrelated** objects. Thus for two objects ***orthogonality*** would imply that the covariance matrix would look as follows:

$\sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$, implying that the covariances are **zero**, and thus the two series are **orthogonal**.

Orthogonality

- The concept of orthogonality can be extended to matrices – where it is defined as: $A^T = A^{-1}$
- Also, if $\boldsymbol{A} =$ **orthogonal**, then it can be shown that $\mathbf{det}(\boldsymbol{A}) = \mathbf{1}$ **or** $\mathbf{-1}$
- All the rows (and columns) of A form an orthonormal set.
  - An **orthonormal** set of vectors implies that all the vectors (rows of A) in the set are mutually orthogonal and all of length one.
  - Thus all the rows can be expressed in normalized form ($\frac{x-\mu}{\sigma}$) as perpendicular vectors of length 1.
  - Or equivalently: **all the rows of A are independent**.

Why is testing for orthogonality useful in econometrics?

- As the law of parsimony in econometrics dictates - we should include as few variables into an equation as possible; thus we should ideally include a set of variables that each explain something completely new. This means if x_1 is saying roughly the same as x_2, we should not include both.

- Thus testing whether $X_1, \ldots X_N$ are all orthogonal, is akin to testing whether all the exogenous variables are linearly independent (and thus all the covariances between them are zero – and thus they are all uncorrelated) → with each adding new information to the regression

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Financial Asset Return data

Stylized Facts about Financial Time-Series data

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Financial asset returns

- Arguably the main objective of financial econometrics in general is to observe asset returns and find out what drives it.
- Asset returns can be defined as the difference between an asset's price at two different time-intervals, and because of the persistence in data – this differencing should remove the unit root.
- Studying financial returns as opposed to asset price levels also removes the scale of the data – allowing cross sector and cross asset comparisons. The approximate stationarity in returns series also allows us to use common statistical techniques. Two types are used:

Simple returns:

$$R_t = \frac{p_t - p_{t-1}}{p_{t-1}} \times 100\%$$

Log Returns:

$$R_t = \ln\left(\frac{p_t}{p_{t-1}}\right) \times 100\%$$

Financial asset returns

- Two stylized facts emerge for most financial asset returns over time:
  - High persistence (requiring First Differencing to remove the unit root)
  - Level increases in volatility.
- Both these factors lead us to the nearly universal treatment of financial time-series: using the **Dlog(X)**.
- Using Log differenced returns is more than just a useful mathematical tool, it has unique interpretation value as well…

Financial asset returns

- Log returns are interpreted as the **continuously compounded returns** of a series.
- It is also useful i.t.o. addition: e.g. from daily returns:
- $R_1 = \ln(p_1) - \ln(p_0)$ ... $R_5 = \ln(p_5) - \ln(p_4)$
- The weekly continuously compounded return is the sum of the daily returns: $R_{week} = \sum_{i=1}^{5} R_i$

  (or annual returns are the sum of the monthly returns, etc.)
- Typically we will find that although asset prices are nearly always non-stationary and time dependent – their returns typically represent a stationary series

Financial asset returns

- Calculating **real** log returns (e.g. Nominal share price returns adjusted for CPI) are also very simple.
- This can be done simply as follows:
- If $P_t = price\ of\ ABSA\ shares$ & $CPI_t = value\ of\ CPI\ index\ at\ t$
- Then the Real price is: $P_{Real} = (P_t / CPI_t) * 100$
- And the one-period **real return** (continuously compounded) is:

$$R_{Real,t} = \ln\left[\frac{P_t}{P_{t-1}} \div \frac{CPI_t}{CPI_{t-1}}\right] = dlog(P_t) - dlog(CPI_t)$$

- Or in simpler notation: $x_t = dlog(X_t) \quad \rightarrow \quad r^{Real} = r_t - \pi_t$

In the practical class, I will ask you to calculate the real return on the JSE ALSI for 2013.

Financial asset returns

- One downside of log returns: Adding a portfolio of returns become problematic.
- To illustrate this, suppose that $R_i = \log return\ of\ asset\ i$.
- If the portfolio has weighting of w_i for each asset's inclusion, then

$$\mathrm{Log}\left(\mathrm{R}_{p,t}\right) = log\left\{1 + \sum_{i=1}^{N} w_i.\left(\frac{\Delta P_t}{P_{t-1}}\right)\right\} \neq \sum_{i=1}^{N} w_i.\, dlog(R_{i,t})$$

- This is because from log notation: $Log(x + y) \neq \log(x) + \log(y)$
- Thus in order to construct portfolio returns, we normally use the simple returns that can be added.

Conditional / Unconditional

- **Conditionality** takes the past information into account.
- **Think back on the ARMA modelling:**
- For $Y_t = persistent\ AR(1)\ process$:
  - Conditional Mean Equation $\rightarrow$ $E_{t-1}(y_t) = \alpha + \beta\, y_{t-1}$
  - Unconditional Moments $\rightarrow$ $E(y_t) = \frac{\alpha}{1-\beta}$; $Var(y_t) = \frac{\sigma^2}{1-\beta^2}$
- Thus we take information from period $t-1$ into account when measuring the conditional mean equation, but for the unconditional mean equation we calculate the long term (constant) mean.
- We will use the concept of conditional moments further…

Stylized Facts: Financial asset returns

- Despite us not being able to really generalize across all sectors and financial assets (e.g. shares, bonds, commodities and derivatives all behave very distinctly), there are some key aspects that are inherent to nearly all **financial asset returns**.

These include the following:

- Heavy (**fat**) **Tail** distributions
- **Asymmetries in return persistence** – negative news cause more pronounced periods of persistence than positive news. See Fischer Black (1976) who explain this in terms of **leverage** effects on the balance sheets of firms – implying simply that the leverage ratio (indebtedness) of a firm increases if the value of the firm decreases

Stylized Facts: Financial asset returns

- **Assumed Distributions** aggregate to log normality in returns when $t \to \infty$ (strong assumption…)
- **Timing matters**: When markets are closed, information accumulates, and is acted upon when it opens again. Thus opening prices, prices after weekends / holidays and prices after big announcements can display far more volatility than normal. This is NB for **high frequency** data (e.g. daily).
- **Volatility clustering**: Empirical data show that most return series display strong persistence in volatility, also known as volatility clustering or momentum. This positive autocorrelation found in second moments of returns (i.e. return variances) imply the presence of ***conditional heteroskedasticity***, even though the series displays homoscedasticity over the longer term (thus being unconditionally homoscedastic)

Stylized Facts: Financial asset returns

- **Long memory in second moments**: Engle, Bollerslev and several others showed that financial asset returns, although showing relatively low memory in mean returns (as a result of market efficiencies) – show long memory in second order moments. This will be returned to when we model fractional integration in the second order moments.
- **Typically reject the JB-Normality tests:**

$$JB = \frac{\frac{SK^2}{6}}{t} + \frac{\frac{EKU^2}{24}}{T} \sim \chi^2(1) + \chi^2(1) \sim \chi^2(2)$$

  - Which is a joint test of the normality of the skewness and kurtosis.

Recap: Moments

- The term, ***moments***, are often used in econometrics.
- Let's consider the first four moments of a **Log Return variable**:

If $X = dlog(Y)$, we typically expect the following four moments (centred) :

$$m_1 = \boldsymbol{mean}\ (\boldsymbol{the\ centre}) = E(X) = \mu = 0$$

$$m_2 = \boldsymbol{variance}\ (\boldsymbol{dispersion}) = Var(Y) = E(X^2) - E(X)^2 = \sigma^2$$

$m_3 = \boldsymbol{SK} = \boldsymbol{skewness} = E[\left(\frac{X-\mu}{\sigma}\right)^3]$ → Asymmetry of the distribution of returns

(The sign indicates whether positive or negative returns are more likely)

$m_4 = \boldsymbol{EKU} = \boldsymbol{Kurtosis} = E[\left(\frac{X-\mu}{\sigma}\right)^4]$ → Thickness of the tails relative to Normal

distribution. Normal distr have $m_4 = 3$, and financial returns typically above 3

Stylized facts of financial asset return data

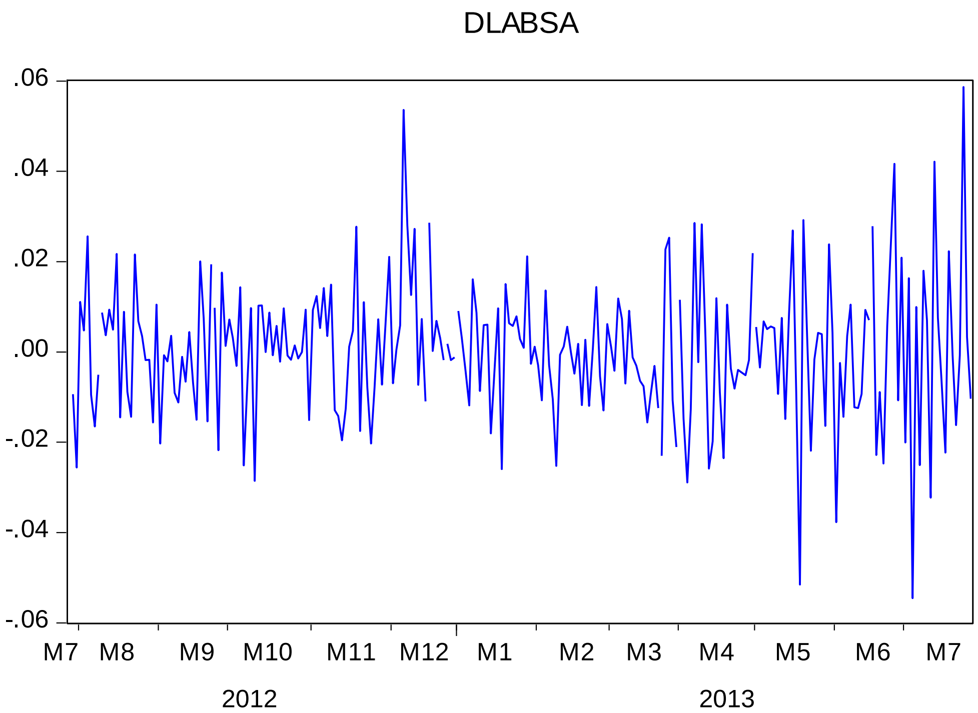
- Looking at $ABSA$'s daily closing prices for the last 12 months.

NB for getting asset prices / indexes

- As most of you will be downloading share price / index data, remember the following crucial point:
  - Shares: The payout of dividends and stock splits distort closing prices, and as such you should consider adjusted closing prices, which takes into account the above and more.
  - Indexes: Similarly, dividends, splits, exercising options, etc. also effect a basket of assets – not reflected in closing prices. This needs to be taken into account when studying index data, and as such researchers often use adjusted closing prices or Total Return Indexes (which is what investors typically are most interested in when holding a portfolio).
  - Now the choice of Gross / Net dividends can be a bit of a contentious choice… see: http://europe.etf.com/europe/publications/journal-of-indexes/articles/7786-dividend-tax-leakage-in-popular-equity-indices.html?showall=&fullart=1&start=5

Stylized facts of financial asset return data

- The data was summarized as follows in:

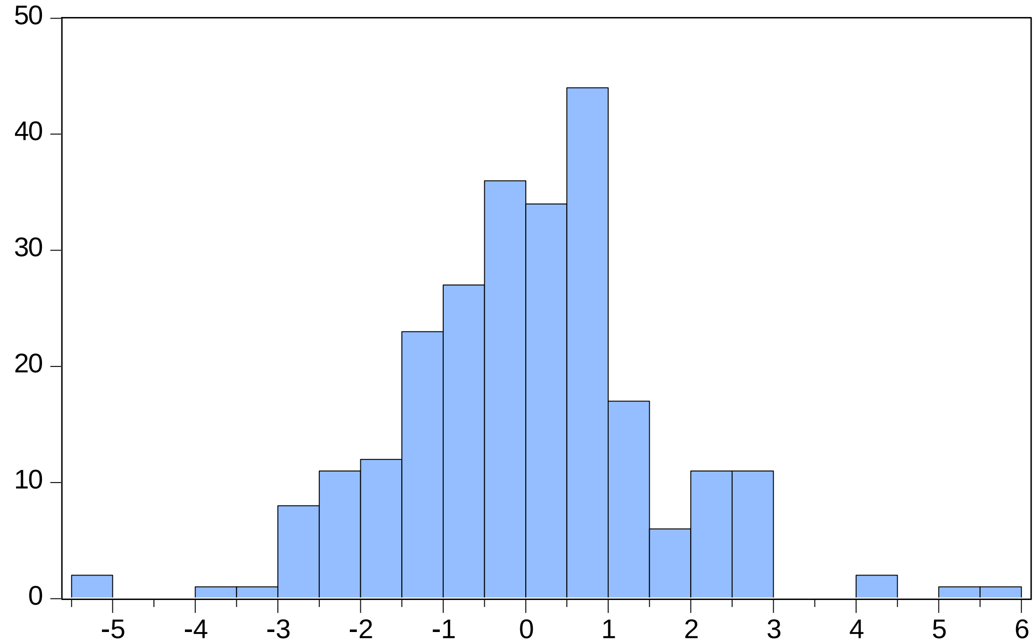


	DLABSA
Mean	0.023136
Median	0.034062
Maximum	5.867170
Minimum	-5.451793
Std. Dev.	1.537987
Skewness	0.138016
Kurtosis	4.606721
Jarque-Bera	27.46337
Probability	0.000001
Sum	5.737711
Sum Sq. Dev.	584.2546
Observations	248

ABSA real daily returns distribution for the year

- From the output above, it is clear that volatility in returns has increased in the latter part of the sample (the banking industry has in the last month seen some dramatic swings in share prices…).
- Also compare this to some of the stylized facts that we discussed:
  - **Skewness** = the share returns are significantly **positively** skewed.
  - It also displays a **Leptokurtic distribution** (**Kurtosis > 3**), implying it has a **fattish tailed distribution** - although not as much as you'd typically see for other returns.
  - It also unsurprisingly strongly rejects the Jarque-Bera test for normality ($p \rightarrow 0$).
- The mean of the daily continuously compounded return is 0.02%, with a std deviation of 1.53% (which is pretty high relative to the mean!!).

- The **Histogram** can also be viewed (and clearly shows the non-normality of returns and the fat tail):



Series: DLABSA
Sample 7/27/2012 7/25/2013
Observations 248

Mean	0.023136
Median	0.034062
Maximum	5.867170
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Std. Dev.	1.537987
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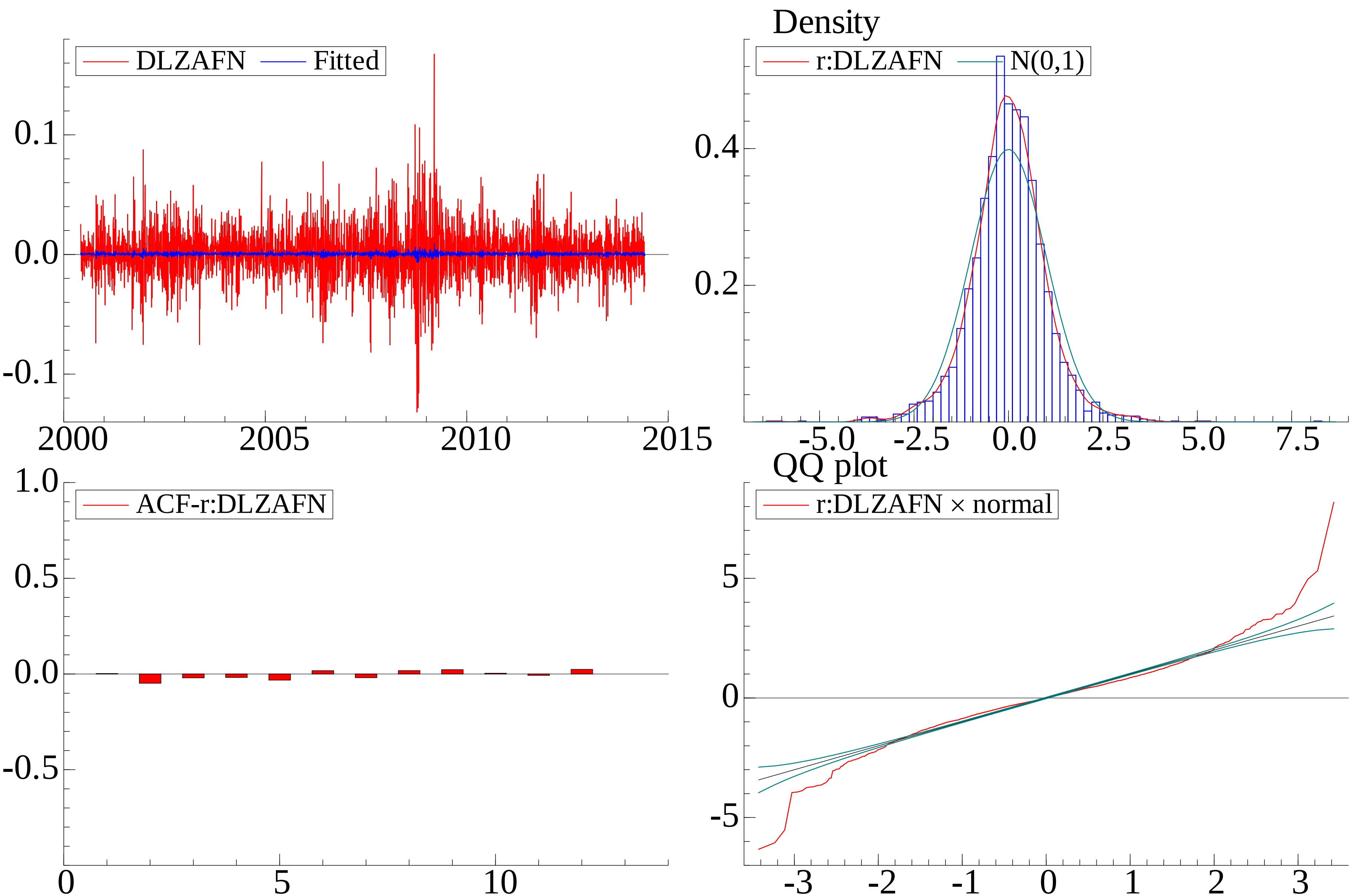
ZA Financials index TRI

- Let's fit an AR(1) term on the Financials index continuously compounded returns, as well as including an intercept (as in the long term the returns are slightly positive, as with ABSA's returns), and then study the closeness of our fit.
  - Most models on asset returns use the most parsimonious and naïve model available when separating noise (ε_t) from the model (μ_t).



Normality test: Chi^2(2) = 977.57 [0.0000]**

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Dependency across assets

Linear dependency across assets

- Up to now we have discussed some stylized facts about individual series.
- Now we discuss some **empirical regularities** for asset co-dependency…
- First of all, remember that the correlation coefficient of $X\&Y$ is:

$$\rho_{X,Y} = \frac{cov(X,Y)}{\sigma_X \sigma_y}$$

Typically these asset price and asset market correlations tend to **increase** in periods of **turmoil** (turmoil leads to asset market price homogeneity), with a negative correlation between assets indicative of the ability to diversify.

Linear dependency across assets

- While **static unconditional correlations** of past returns might give us an indication of the **general** diversification potential of different assets / markets (think Markowitz theories), the need to understand how conditional correlation changes over time and what drives these changes are vital for researchers and practitioners alike.

- This will be returned to in more detail in the Multivariate GARCH sections, where we will study **time-varying conditional correlations**.

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Finding inspiration for your research paper…

When reading the literature on finance

1. Does the paper involve the development of a theoretical model or is it merely a technique looking for an application, or an exercise in data mining? ***– Remember check for other techniques that do the same thing, and ask yourself whether this technique is best for your question.***
2. Is the data of "good quality"? Is it from a reliable source? Is the size of the sample sufficiently large for asymptotic theory to be invoked? ***– Remember to check the data first!***
3. Have the techniques been validly applied? Have diagnostic tests for violations of been conducted for any assumptions made in the estimation of the model?

When reading the literature on finance

4. Have the results been interpreted sensibly? Is the strength of the results exaggerated? Do the results actually address the questions posed by the authors? ***– Often techniques are validly used, but the interpretations are unclear. Remember financial research is not an exercise of mathematical ability, but rather of practical use.***
5. Are the conclusions drawn appropriate given the results, or has the importance of the results of the paper been overstated? ***– ALWAYS be critical of your own work, and don't be afraid to***